Instantaneous Rate of Change (Section 2.1) **Date:** ___________

Last class period, we looked at the problem where a ball is thrown up in the air. The height, y, of the ball above the ground first increases, and then decreases. It's height t seconds after it is thrown is shown in the chart below.

t (sec)	0	0.5	1	1.5	2	2.5	3	3.5	4
y (feet)	5	36	59	74	81	80	71	54	29

Our objective was to calculate the average rate of change (average velocity) of the ball over an interval.

Example: Calculate the average rate of change of the ball on the interval from $t = 0.5$ to $t = 3.5$.

A new question: What is the velocity of the ball at exactly 1.5 seconds? This would be the instantaneous velocity at $t = 1.5$.

We can use the **average velocities** to estimate the **instantaneous velocity**. How?

Suppose we knew the function rule that generated the values in the chart above. Enter the data into the lists of your calculator and create a scatterplot.

What type of function best fits this data? ______________________

Then using the data, perform a **quadratic regression** (QuadReg under STAT, CALC) to find the best fitting quadratic function for the data.

y = ______________________________

How could we use this function to find a better estimate for the instantaneous velocity at $t = 1.5$ seconds?

From the calculations above, it appears that the instantaneous velocity at $t = 1.5$ seconds is exactly __________
This value is the "limiting value" or the "limit" of the average velocities over smaller and smaller intervals.

Definitions:

1. The **instantaneous velocity** of an object at time t is defined to be ______________________________

____

2. The **instantaneous rate of change of a function f at a**, also called the rate of change of f at a, is defined to be

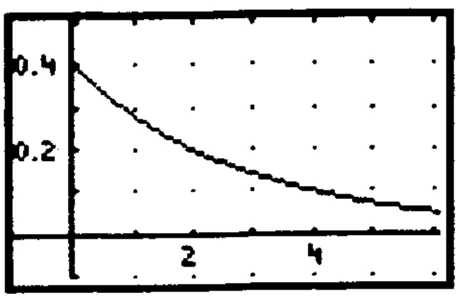
____

____

Example:

Recall the **Nicotine Problem** from a few days ago.

Nicotine Problem: The graph shown below represents the amount of nicotine, $N = f(t)$, in mg, in a person's bloodstream as a function of the time t, in hours, since the person finished smoking a cigarette.



From the graph, we determined that the function rule that represented the problem was: ____________________

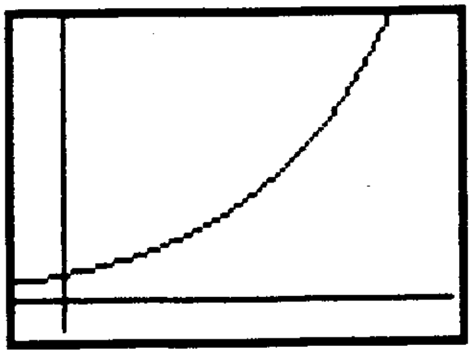
Use the function rule to estimate the instantaneous rate of change of N with respect to t when t = 1. ____________

The **instantaneous rate of change** of a function is so important, that it is given it's own name.

> The **derivative of f at a,** written *f'(a)* (read "f-prime of a), is defined to be the **instantaneous rate of change of f at the value x = a.**

Example: If $\mathbf{f(x) = x^3}$, estimate $\mathbf{f'(2)}$.

Visualizing the Derivative Graphically



Notes:

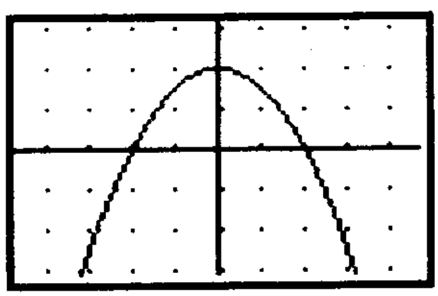
The **average rate of change** of a function is represented by the slope of ______________________

The **instantaneous rate of change** of a function is represented by the slope of ______________________

> The following three statements are equivalent concerning a function f and an x-value of x = a.
>
> 1. ______________________
> 2. ______________________
> 3. ______________________

Example:

The graph of $f(x) = -\frac{1}{2}x^2 + 2$ is shown to the right.



1. Use the graph to determine whether each of the following quantity is positive (+), negative (-), or zero (0).

   a. f(1) ________ b. f(-3) ________ c. f(0) ________ d. f'(1) ________

   e. f'(-3) ______ f. f'(0) ________ g. $\dfrac{f(1)-f(-1)}{1-(-1)}$ ________

2. Which is larger, f(-3) or f(-1)? ____________

3. Which is larger, f'(-3) or f'(-1)? ____________

4. Estimate f'(2) graphically and numerically. Graphically: ________ Numerically: ________

The Derivative Function Graphically (**Section 2.2**) **Date** ____________

In the last class, we looked at the derivative of a function at a point. In general, the derivative takes on different values at different points and is itself a function.

Remember, **the derivative f'(a) is the slope of the tangent line to the graph of f at x = a.**

One question in this lesson is:

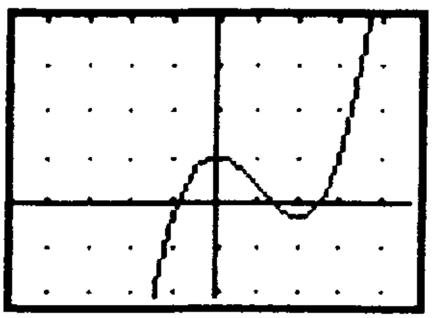
Given the graph of a function f, can we sketch a graph of the derivative function f'?

And another is:

What does the derivative f' tell us about f?

We are going to try to answer these questions, because, if we can, we should have a good understanding of the derivative function!

Example: Below is the graph of the function $f(x)=\frac{1}{3}x^3-x^2+1$.



1. At what x-value(s) does it appear that the derivative of f is zero? x = ____________

2. From the graph, estimate the value of f'(1). ____________ f'(-1) ____________

3. Built into your calculator is a feature that will estimate the derivative of a function at a value. From the Home screen of your calculator, press the MATH key, and select **8:nDeriv(**. (This stands for a "numerical derivative".) The parameters for "nDeriv" are:

nDeriv(function , variable , value)

So, if we want the derivative of $f(x)=\frac{1}{3}x^3-x^2+1$, at x = 0, we would enter:

____

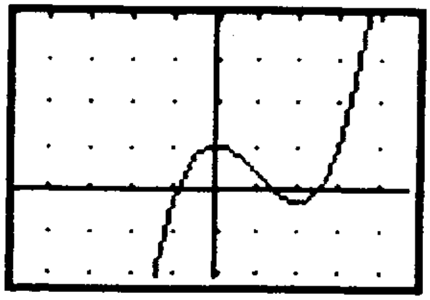
Check your other answers from parts 1 and 2 above, and then complete the chart below.

x	-2	-1	0	1	2	3
f'(x)						

An important point to notice is that for every x-value, there is a corresponding value of f'(x). The derivative, therefore, is also a function of x!

> For a function f, we define the **derivative function,** f', as the instantaneous rate of change of f at x.

On the graph of the function $f(x)=\frac{1}{3}x^3-x^2+1$ below, plot the points (x , f'(x)) that you can fit on the graph from the table on the previous page. Then connect the points with a smooth curve.



Can you guess the rule for the derivative function from its graph?

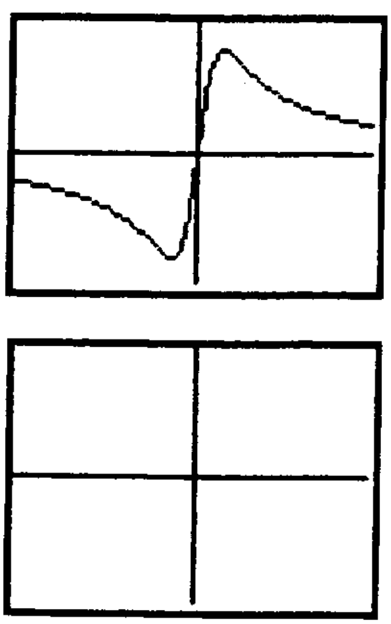
Guess: f'(x) = ____________________

What properties can we write about the relationship between the graph of f and the graph of f'?

1. Where f has a "turning point", ____________________
2. Where f is increasing, ____________________
3. Where f is decreasing, ____________________

Examples: Given the graph of the function shown, sketch the graph of the derivative function f' directly below it. Remember, **"the y value on the graph of f' is the slope of the tangent line to the graph of f."**

1.



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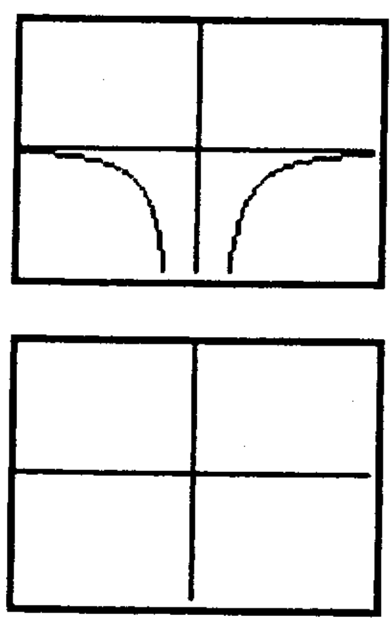
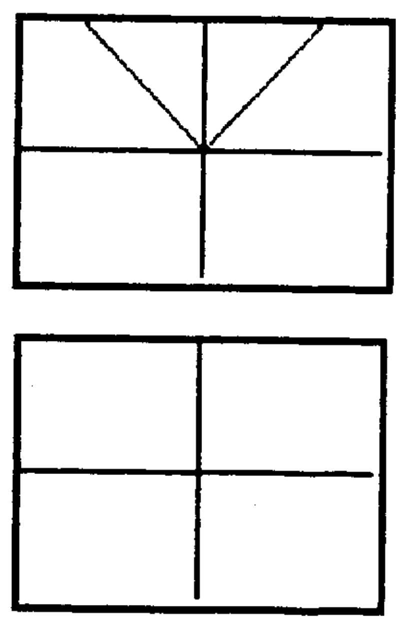


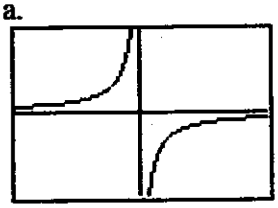
3. .



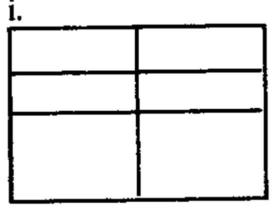
Graphs of f and f'

1. In the left column below are graphs of several functions. In the right-hand column - in a different order - are graphs of the associated derivative functions. Match each function with its derivative. (Note: The scales on the graphs are not all the same.)

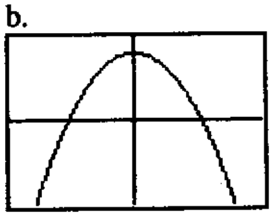
a.



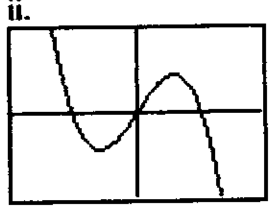
i.



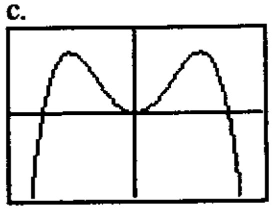
b.



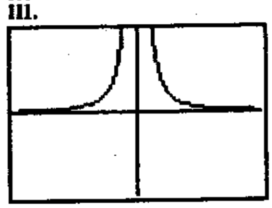
ii.



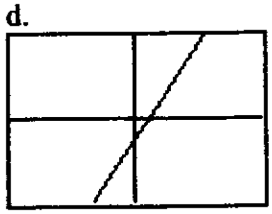
c.



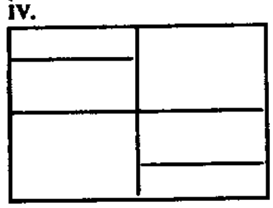
iii.



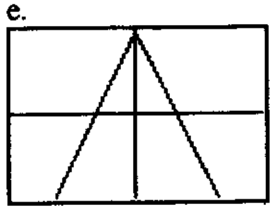
d.



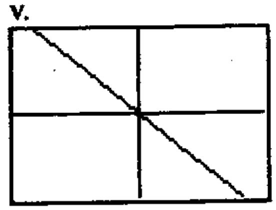
iv.



e.



v.



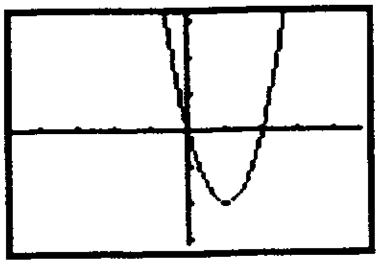
2.(a) Sketch a graph of the derivative of each function labeled (i) - (v) in the right column of the preceding problem.

(b) (Optional!) For each function labeled (a) - (e) in the left column of the preceding problem, sketch a graph of a function whose derivative is the function shown.

The second question we wanted to answer was: **What does the derivative f' tell us about f?**

Examples: Below is the graph of f', the derivative of a function f.

1.



On what interval(s) is the function f

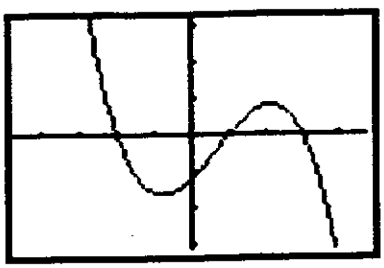
a. increasing? ____________________

b. decreasing? ____________________

At what x-value does f have a

c. maximum? ________ d. minimum? ________

2.



On what interval(s) is the function f

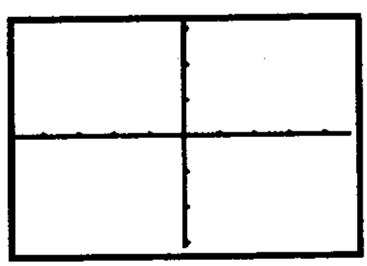
a. increasing? ____________________

b. decreasing? ____________________

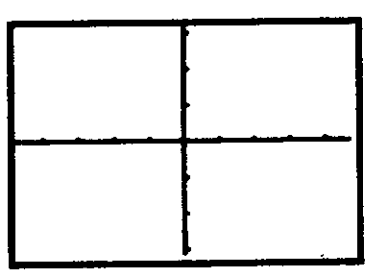
At what x-value does f have a

c. maximum? ________ d. minimum? ________

3. Draw a possible graph of $y = f(x)$ given the following information.
   - $f(1) = 2$
   - $f'(x) > 0$ on $x < 1$
   - $f'(x) < 0$ on $x > 1$



4 Draw a possible graph of $y = f(x)$ given the following information.
   - $f(-2) = 0$
   - $f'(0) = 0$
   - $f'(x) < 0$ on $x < 0$
   - $f'(x) < 0$ on $x > 0$



Interpretations of the Derivative (Section 2.3) **Date** ______________

There is an alternative notation for the derivative that we have to be familiar with.

We know that the derivative $f'(x)$ is approximated by the <u>average rate of change over a small interval</u>. Therefore, if $y = f(x)$, then the average rate of change is given by $\Delta y / \Delta x$. And, for "small" Δx, we have:

$$f'(x) \approx \frac{\Delta y}{\Delta x}.$$

To remind us of this, if $y = f(x)$, we can write; $f'(x) = \dfrac{dy}{dx}$.

This is known as writing the derivative using "Leibniz notation".

A disadvantage of this notation is how we need to write the "derivative evaluated at a number". Using this alternative notation, in order to write $f'(5)$, we have to write $\left.\dfrac{dy}{dx}\right|_{x=5}$.

Example: Recall that if $s = f(t)$ represents the position of a moving object at time t, then $v = f'(t)$ is the velocity of the object at time t. Using Leibniz notation we can write:

$v =$ ____________________ and $f'(3) =$ ______________________

Also, since the "derivative of velocity" is the "rate of change of velocity", then $\dfrac{dv}{dt}$ represents ____________________________________. If the units for the velocity are meters/sec, then the units for acceleration would be: ____.

So, a new and important question! **What is the meaning of the derivative in a real world situation?** We will try to answer this question through some examples.

Examples:

1. The population of the world, P in billions of people, is a function of the year t. Therefore, $P = f(t)$. Explain:

   a. $f(1990) = 5.295$

   b. $f'(1990) = 0.086$

   c. Use this information to estimate the world population in 1991.

2. The cost, C (in dollars) to produce g gallons of ice cream can be expressed as $C = f(g)$. Using units, explain the meaning of the following statements in terms of ice cream.

a. $f(200) = 350$

b. $f'(200) = 1.4$

c. Estimate $f'(199)$

3. The table below shows world gold production, $G = f(t)$, as a function of the year, t.

t (year)	1990	1993	1996	1999	2002
G (mn troy ounces)	70.2	73.3	73.6	82.6	82.9

a. Does $f'(t)$ appear to be positive or negative? __________ This is because the gold production is ____________________

b. In which time interval does $f'(t)$ appear to be the greatest? ____________________

c. Estimate $f'(2002)$. Give units and interpret your answer in terms of gold production.

d. Use the estimated value of $f'(2002)$ to estimate $f(2003)$ and $f(2010)$.

$f(2003) \approx$ __________ $f(2010) \approx$ __________

4. The table below shows a function $f(t)$, the total sales of music compact discs (CDs), in millions.

t (year)	1994	1996	1998	2000	2002
CD sales	661.2	778.9	847.0	942.5	803.3

a. On what interval(s), does $f'(t)$ appear to be positive? __________ negative? __________

b. Estimate $f'(2002)$. Give units and interpret your answer your answer.

c. Use the estimated value of $f'(2002)$ to estimate $f(2003)$ and $f(2010)$.

$f(2003) \approx$ __________ $f(2010) \approx$ __________

The Second Derivative (Section 2.4)

Date ________________

Since the derivative is itself a function, we can calculate its derivative. This is called the **second derivative** and, as we will see, it also gives us useful information about the original function.

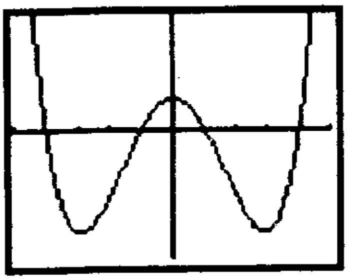
First, let's look at some notation. Let $y = f(x)$.

First derivative: Second derivative:

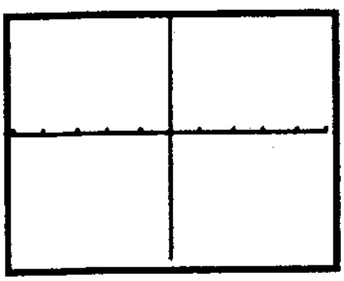
Question: What information does the second derivative tell us?

Let's look at this graphically with an example.

This is the graph of a function f.



Sketch the graph of the derivative of f, f'.

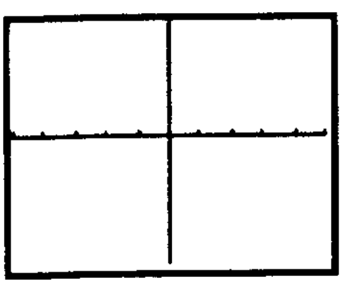


Recall the information that the derivative tells us about a function.

On an interval,

- if $f' > 0$, then ______________________________
- if $f' < 0$, then ______________________________

Sketch the graph of the derivative of f', f''.



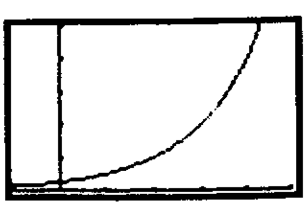
Since f'' is the derivative of f', we have:

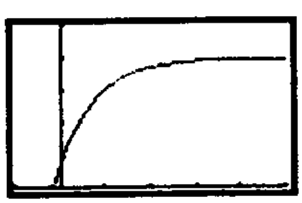
On an interval,

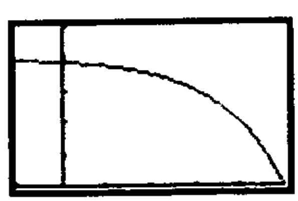
- if $f'' > 0$, then ______________________________
- if $f'' < 0$, then ______________________________

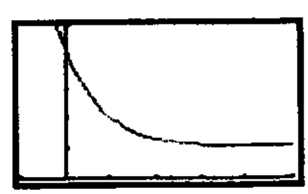
So, the question now becomes: **What does it mean for f' to be increasing or decreasing?**

Look at the graphs of four functions below. For each graph, determine if f' is **increasing** or **decreasing**? Remember, f' represents the <u>slope of the curve</u> (or the <u>slope of the tangent line</u>).









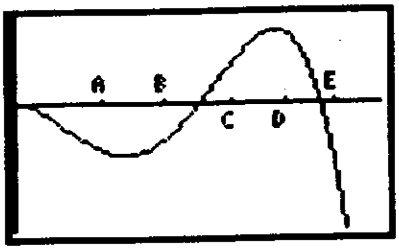
f' is ____________	f' is ____________	f' is ____________	f' is ____________
f is ____________	f is ____________	f is ____________	f is ____________

Conclusions: On an interval,

$f'' > 0$ means f' is ____________________, so f is ____________________
$f'' < 0$ means f' is ____________________, so f is ____________________

Examples:

1. Given the graph of a function f below. At which marked x-value(s) are the following statements true?



a. $f(x) < 0$ __________ d. $f(x)$ is inc __________

b. $f'(x) < 0$ __________ e. $f'(x)$ is inc __________

c. $f''(x) < 0$ __________

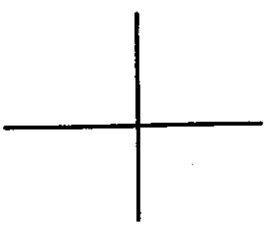
2. Let P(t) represent the price of a share of stock of a corporation at time t. What do each of the following statements tell us about the signs of the first and second derivatives of P(t)?

a. " The price of the stock is rising faster and faster". $P'(t)$ ________ $P''(t)$ ________

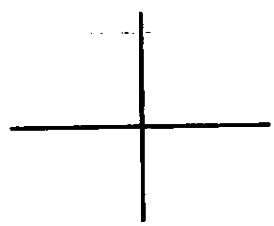
b. "The price of the stock is close to bottoming out". $P'(t)$ ________ $P''(t)$ ________

3. Sketch a graph of a continuous function with the following properties.

a. $f(0) = 1,\ f'(0) = -2,\ f''(0) < 0$



b. $f(0) = -1,\ f'(0) = 1,\ f''(-1) > 0, f''(1) < 0$



Limits and the Derivative Function **Date:** ____________

Earlier we defined $f'(a)$, the derivative at $x = a$, graphically as:

An <u>alternative definition</u> for $f'(a)$ could be:

Both of these definitions rely on an understanding of a **limit**. So let's develop the concept of a limit. It's important that we understand this concept!

Examples

1. Let $f(x) = 2x + 1$. As x gets closer and closer to some number, say 3, does $f(x)$ get closer and closer to some value L? If it does, then we write $\lim_{x\to 3}(2x+1) = L$.

Let's see. Evaluate: $f(2.9) =$ ______ $f(2.99) =$ ______ $f(2.999) =$ ______

$f(3.1) =$ ______ $f(3.01) =$ ______ $f(3.001) =$ ______

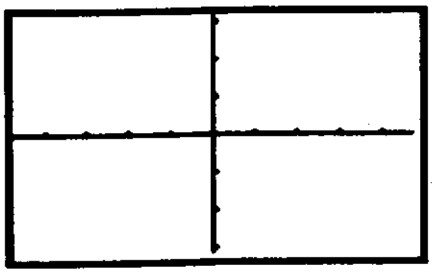
So, the $\lim_{x\to 3}(2x+1) =$ _____

How else could we have evaluated this limit?

2. Find $\lim_{x \to 2} \frac{x-2}{x^2-2x}$.   Can we find the limit by substituting 2 for x? __________ Explain!

Let's look at this problem graphically. Graph the function $f(x) = \frac{x-2}{x^2-2x}$ on your calculator.

Use a "Decimal window" by selecting ZOOM, 4:ZDecimal.



What do you notice about the graph?

____

Evaluate the function close to $x = 2$ to determine the limit.   Conclusion: $\lim_{x \to 2} \frac{x-2}{x^2-2x} =$ ________

How could we have evaluated this limit analytically?

$$\lim_{x \to 2} \frac{x-2}{x^2-2x} =$$

Examples:

1. Graphically, find the following limit. Write your answer to 3 decimal places.
   (Note: The answer to this question is a very important number. Do you remember what it is?)

   $\lim_{x \to 0} (1+x)^{1/x} =$ ____________________

2. Analytically, find the following limit.

$$\lim_{x \to -3} \frac{x^2-9}{2x^2+5x-3}$$

Let's go back to our new definition of a derivative for the next example.

Definition: $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

Example: If $f(x) = 2x^2 + 3x$, find $f'(-2)$.

Practice Problem: If $f(x) = -x^2 - 4$, use the definition of the derivative to show that $f'(-3) = 6$.

Since we know that the derivative of a function f(x) is another function f'(x), we can find this derivative function if we substitute the variable **x** for the value **a** into the derivative definition. This gives us the **definition of the derivative function f'(x).**

Given a function **f(x)**, the derivative **f'(x)** is defined to be:

f'(x) =

Example: If $f(x) = 2x^2 + 3x$, use the definition of the derivative to find f'(x).

Practice Problem: If $f(x) = -x^2 - 4$, use the definition of the derivative to show that $f'(x) = -2x$.

Example: If $f(x)=\frac{1}{x^2}$, use the definition of the derivative to find $f'(x)$.

Practice Problem: If $f(x)=\frac{1}{x}$, use the definition of the derivative to show that $f(x)=-\frac{1}{x^2}$.